PMM U.S.S.R., Vol.48, No.4, pp. 380-387, 1984
Printed in Great Britain
0041-8928/84 $10.00+0.00

THE DYNAMICS OF RHEONOMIC LAGRANGIAN SYSTEMS WITH CONSTRAINTS*

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General rheonomic Lagrangian systems with non-integrable constraints are considered. Their equations of motion, in the form of the Lagrange equations with multipliers, are equivalent to the variational equation expressing the Hamilton principle in Hölder form, which occurs when determining the synchronous Chetaev variations. A parametric study of the motions of such systems is made in the extended configurational space R_{n+1} with arbitrarily chosen parameters. The virtual displacements Δq_i in the space R_{n+1} represent the total (asynchronous) variations for the configurational space R_n. The Hamilton principle with homogeneous Lagrangian is used to derive $n+1$ parametric equations of motion with multipliers, one of which follows from the remaining n multipliers. When the time t is chosen as a parameter, the equations take the form of the usual equations with multipliers.

When the Lagrangian is independent of t and the constraints are homogeneous with respect to the velocities, the equations of motion have an energy integral corresponding to the ignorable coordinate t. Elimination of t' from the Hamilton principle leads to the principle of least action in the Jacobi or Lagrangian form. The energy integral is used to reduce the order of the initial equations, and a generalization of the Jacobi-Whittaker equations is obtained as a result. Finally, the problem of two forms of the theorem on energy is discussed for the usual dynamic rheonomic systems with constraints.

1. Let us consider a general rheonomic Lagrangian system with non-integrable constraints characterized by the Lagrange function $L(q, t, q^{\cdot})$ and perfect indepdent constraints of the form

$$f_l(q, t, q^{\cdot}) = 0 \quad (l = 1, \ldots, r) \tag{1.1}$$

in general, non-linear in $q_i^{\cdot} \equiv dq_i/dt$. Here, if we use the mechanical terminology, /1/ q_i $(i = 1, \ldots, n)$ are the independent Lagrangian coordinates, t is the time and $q_i^{\cdot}$ are the generalized velocities. We assume that the functions $L(q, t, q^{\cdot}) \in C^2$, $f_l(q, t, q^{\cdot}) \in C^2$ are defined at all points of some fixed, simgly connected region G of the space R_{2n+1} of variables q_i, t, $q_i^{\cdot}$, where

$$\det\left(\frac{\partial^2 L}{\partial q_i^{\cdot} \partial q_j^{\cdot}}\right) \neq 0, \quad \operatorname{rank}\left(\frac{\partial f_l}{\partial q_i^{\cdot}}\right) = r$$

The equations of motion of the general rheonomic Lagrangian system with constraints have the form of the Lagrange equations with undetermined coefficients μ_l

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} = \sum_{l=1}^{r} \mu_l \frac{\partial f_l}{\partial q_i^{\cdot}} \quad (i = 1, \ldots, n) \tag{1.2}$$

which must be supplemented by the constraint equations (1.1). The general solution of the system of equations (1.1), (1.2) depends on $2n - r$ arbitrary constants determined by specifying the initial data. Equations (1.2) can be obtained from the Hamilton principle in the Hölder form

$$\int_{t_0}^{t_1} \delta L\, dt = 0, \quad \delta q_i = 0: \ t = t_0, t_1 \tag{1.3}$$

The symbol δ denotes the isochronous variation (for $\delta t = 0$) i.e. the variation on the virtual displacement, and the virtual displacements $\delta q_i(t) \in C^2$ under the constraints (1.1) satisfy the Chetaev /2/ conditions

*Prikl.Matem.Mekhan., 48, 4, 540-550, 1984

$$\sum_{i=1}^{n} \frac{\partial f_l}{\partial q_i^{\cdot}} \delta q_i = 0 \quad (l = 1, \ldots, r) \tag{1.4}$$

The actual trajectory written in the parameteric form $q_i = q_i(t)$ satisfying the equations (1.1) and passing through two fixed points $P_0(q_i^\circ)$ and $P_1(q_i^1)$ of the configurational space R_n at fixed instants of time $t_0 < t_1$ respectively, is compared in (1.3) with the similar curves $q_i = q_i(t) + \delta q_i$. The curves also connect the points $P_0$and P_1, and the time of motion of the system between them along all comparison curves do not, in general, satisfy (1.1), when the constraints are non-integrable. This implies that the Hamilton principle (1.3) does not represent, in the case of a non-holonomic system, a variational principle in the sense of variational computation, but merely a variational equation. Also, as was shown before in /3/, the equations of motion (1.2) are in general not equivalent to the Euler-Lagrange equations for the variational Lagrange problem.

In this connection we note the incorrectness of the assertion /4, 5/ that the equations in question are equivalent in the case of the constraints homogeneous in $q_i^{\cdot}$. The equation

$$\sum_{i=1}^{n} \left(\frac{\partial f_l}{\partial q_i} - \frac{d}{dt} \frac{\partial f_l}{\partial q_i^{\cdot}} \right) q_i^{\cdot} = 0$$

does not generally imply, notwithstanding /4/, the relation

$$\frac{\partial f_l}{\partial q_i} = \frac{d}{dt} \frac{\partial f_l}{\partial q_i^{\cdot}}$$

Conversely, we can obtain the principle (1.3) from (1.2), taking (1.4) into account. To do this, we multiply (1.2) by δq_i, carry out the summation over all i, take (1.4) into account and integrate in t from t_0 to t_1. From this it follows, that we have, for the general Lagrangian systems with constraints, full equivalence /1/ between the equations (1.2), (1.1) and variational equation (1.3) when deriving the virtual displacements from (1.4).

We use the time t as the independent variable in (1.2), (1.3). The time plays a special role in the processes which take place in the configurational space $R_n$in time t. Generally, the equations (1.2) and (1.3) are invariant with respect to the point transformations of the Lagrangian coordinates q_i, but not invariant with respect to the transformations of the variable t.

Problem (1.3) can obvioulsy be considered in an extended configurational space R_{n+1}, whose point coordinates are q_i, t and where the motion is represented by a curve $q_i(t)$ specified now in the explicit, and not the parametric form. Here the integral in (1.3) is taken over all possible curves close to comparison curves, connecting two fixed points (q_i°, t_0) and (q_i^1, t_1) of the space R_{n+1}. In both spaces, R_n and R_{n+1}, the number of equations (1.2), when t is used as the independent variable, is equal to the number n of the Lagrange coordinates q_i. Together with (1.1), these equations fully describe the dyhamics of Lagrangian systems, both scleronomic and rheonomic. The reactions of the constraints (1.1) are also found.

In variational computations, it is often more convenient to specify the comparison curves not in explicit, but in parametric form, with arbitrarily chosen parameter /6/. In the dynamics of holonomic systems the parametrization enables us to show the close connection between various variational principles /7/. The same problem is also of interest in the case of non-holonomic systems /8/.

We shall therefore consider the time t together with Lagrangian coordinates q_i as the equivalent and independent variables representing the coordinates of the points belonging to the space R_{n+1}. We shall denote these variables by x_α $(\alpha = 1, \ldots, n+1)$ with $x_i = q_i$ $(i = 1, \ldots, n)$, $x_{n+1} = t$. All these variables can be specified as continuous differentiable functions of some parameter τ, chosen as an arbitrary function $\tau(t) \in C^1$ with $d\tau/dt > 0$ for all values of t considered. The choice of the parameter has no special significance. We can replace the parameter τ by any other parameters σ, provided that $d\sigma/d\tau > 0$.

Suppose $x_\alpha = x_\alpha(\tau)$ are curves belonging to the class $C^2 \in R_{n+1}$ such, that $x_\alpha' \equiv dx_\alpha/d\tau$, are not simultaneously zero for any value of τ. The curves, with certain specified directions along them, correspond to the possible motions of the holonomic system without constraints (1.1). However, when such constraints exist, which, in the variables x_α, take the form

$$F_l(x_\alpha, x_\alpha') = 0 \quad (l = 1, \ldots, r) \tag{1.5}$$

then not all curves $x_\alpha = x_\alpha(\tau)$ will describe the possible motions of the system, but only those curves for which the variables x_α, x_α' satisfy the conditions (1.5). Here $F_l(x_\alpha, x_\alpha')$ are functions defined by the equations

$$F_l(x_\alpha, x_\alpha') = f_l\left(x_1, \ldots, x_{n+1}, \frac{x_1'}{x_{n+1}'}, \ldots, \frac{x_n'}{x_{n+1}'}\right) \quad (l = 1, \ldots, r) \tag{1.6}$$

The functions do not depend explicitly on τ, and represent homogeneous functions of zero power in x_α', so that

$$\sum_{\alpha=1}^{n+1} \frac{\partial F_l}{\partial x_\alpha'} x_\alpha' = 0 \quad (l = 1, \ldots, r) \tag{1.7}$$

By varying (1.6) we obtain

$$\frac{\partial F_l}{\partial x_i} = \frac{\partial f_l}{\partial q_i}, \quad \frac{\partial F_l}{\partial x_i'} = \frac{1}{t'} \frac{\partial f_l}{\partial q_i^{\cdot}} \quad (i = 1, \ldots, n) \tag{1.8}$$
$$\frac{\partial F_l}{\partial x_{n+1}} = \frac{\partial f_l}{\partial t}, \quad \frac{\partial F_l}{\partial x_{n+1}'} = -\frac{1}{t'} \sum_{i=1}^{n} q_i^{\cdot} \frac{\partial f_l}{\partial q_i^{\cdot}}$$

Let us assume that when the constraints (1.5) hold, then the virtual displacements Δx_α satisfy the Chetaev-type conditions

$$\sum_{\alpha=1}^{n+1} \frac{\partial F_l}{\partial x_\alpha'} \Delta x_\alpha = 0 \quad (l = 1, \ldots, r) \tag{1.9}$$

Taking (1.8), into account we can write conditions (1.9) in the form

$$\sum_{i=1}^{n} \frac{\partial f_l}{\partial q_i^{\cdot}} (\Delta q_i - q_i^{\cdot} \Delta t) = 0 \tag{1.10}$$

Equating (1.10) and (1.4) we obtain

$$\Delta q_i = \delta q_i + q_i^{\cdot} \Delta t \tag{1.11}$$

which mean that the virtual displacements Δq_i and R_{n+1} represent, for the space R_n, the complete (asynchronous) variations.

When conditions (1.10) hold, we know /9/ that we can generalize the Hamilton principle (1.3) to the case of asynchronous variations and write it in the form of the Foss principle

$$\Delta \int_{t_0}^{t_1} L\, dt = 0, \quad \Delta q_i = \Delta t = 0: \ t = t_0, t_1 \tag{1.12}$$

i.e. the Hamilton principle also holds for the general Lagrangian systems with constraints in the case of asynchronously varied motions, provided that the motions take place between the same configurations and over the same time interval /1/.

In the case of the constraints (1.1), homogeneous in $q_i^{\cdot}$ and satisfying, in accordance with Euler's theorem on homogeneous functions, the conditions

$$\sum_{i=1}^{n} \frac{\partial f_l}{\partial q_i^{\cdot}} q_i^{\cdot} = k_l f_l(q, t, q^{\cdot}) = 0 \quad (l = 1, \ldots, r) \tag{1.13}$$

relations (1.10) by virtue of (1.1), take the form of the conditions

$$\sum_{i} \frac{\partial f_l}{\partial q_i^{\cdot}} \Delta q_i = 0$$

which are identical with (1.4); k_l denotes the degree of homogeneity in $q_i^{\cdot}$ of the functions $f_l(q, t, q^{\cdot})$. Thus we find that when the constraints (1.1) are homogeneous in $q_i^{\cdot}$, the class of synchronous variations (virtual displacements) δq_i is equivalent to the class of asynchronous variations.

Using the given Lagrange function $L(q, t, q^{\cdot})$, we determine the homogeneous Lagrangian for the space R_{n+1} in parameteric form, by the following equation /7/:

$$\Lambda(x_\alpha, x_\alpha') = L\left(x_1, \ldots, x_{n+1}, \frac{x_1'}{x_{n+1}'}, \ldots, \frac{x_n'}{x_{n+1}'}\right) x_{n+1}' \tag{1.14}$$

The function $\Lambda(x_\alpha, x_\alpha')$ does not depend explicitly on τ and is clearly a positively homogeneous first-degree function in x_α', so that

$$\sum_{\alpha=1}^{n+1} \frac{\partial \Lambda}{\partial x_\alpha'} x_\alpha' = \Lambda(x_\alpha, x_\alpha') \tag{1.15}$$

Conversely, using the given homogeneous Lagrangian $\Lambda(x_\alpha, x_\alpha')$, we obtain the following Lagrange function not restricted by a similar condition:

$$L(q, t, q^{\cdot}) = \Lambda(q_1, \ldots, q_n, t, q_1^{\cdot}, \ldots, q_n^{\cdot}, 1)$$

Thus the functions $\Lambda(x_\alpha, x_\alpha')$ and $L(q, t, q^\cdot)$ are equivalent to each other in the sense that one defines the other. This implies the invariance of the element of Lagrangian action /7/

$$L(q, t, q^\cdot)\, dt = \Lambda(x_\alpha, x_\alpha')\, d\tau$$

in which case the Hamilton principle (1.3), on parametrization, takes the form

$$\int_{\tau_0}^{\tau_1} \Delta\Lambda(x_\alpha, x_\alpha')\, d\tau = 0, \quad \Delta x_\alpha = 0: \ \tau = \tau_0, \tau_1 \tag{1.16}$$

with conditions (1.9). The value of the functional on the left-hand side of equation (1.16) depends, as we know, /6/, only on the curve $x_\alpha = x_\alpha(\tau)$ in the space R_{n+1}, and not on the functions $x_\alpha(\tau)$ themselves.

The principle (1.16), together with (1.9), yield the parameteric equations of motion of a non-holonomic system in the space R_{n+1}

$$\frac{d}{d\tau}\frac{\partial\Lambda}{\partial x_\alpha'} - \frac{\partial\Lambda}{\partial x_\alpha} = \sum_{l=1}^{r} \mu_l \frac{\partial F_l}{\partial x_\alpha'} \quad (\alpha = 1, \ldots, n+1) \tag{1.17}$$

where μ_l are undetermined multipliers. Combining equations (1.17) with the constraint equations (1.5), we obtain a system of $n+1+r$ equations with the same number of variables x_α, μ_l. However, equations (1.17), whose number is equal to the number of variables x_α, are not independent, but connected, as in the case of holonomic systems /7/, by the identity

$$\sum_{\alpha=1}^{n+1} x_\alpha' \left(\frac{d}{d\tau}\frac{\partial\Lambda}{\partial x_\alpha'} - \frac{\partial\Lambda}{\partial x_\alpha} - \sum_{l=1}^{r} \mu_l \frac{\partial F_l}{\partial x_\alpha'}\right) \equiv 0$$

which is obvious by virtue of the homogeneity of the functions $\Lambda(x_\alpha, x_\alpha')$, $F_l(x_\alpha, x_\alpha')$ in x_α' and their independence of τ.

The last equation of (1.17) follows from the first n equations. Indeed, if we multiply the first n equations by x_i' and sum over $i = 1, \ldots, n$, then taking (1.7) and (1.15) into account, we obtain the equation

$$-x_{n+1}'\left(\frac{d}{dt}\frac{\partial\Lambda}{\partial x_{n+1}'} - \frac{\partial\Lambda}{\partial x_{n+1}} - \sum_l \mu_l \frac{\partial F_l}{\partial x_{n+1}'}\right) = 0$$

from which, putting $x_{n+1}' \neq 0$ we obtain the last $(\alpha = n+1)$ equation of (1.17). Consequently, the number of independent equations of motion in the parameteric form (1.17) is equal to the number of independent Lagrangian coordinates q_i. The general solutions of the equations (1.17), (1.5) depend on $2n - r$ arbitrary constants.

Let us equate the equations of motion (1.2) with the parameteric equations (1.17).Varying equations (1.14) we easily establish the relations

$$\frac{\partial\Lambda}{\partial x_i} = t'\frac{\partial L}{\partial q_i}, \quad p_i \equiv \frac{\partial\Lambda}{\partial x_i'} = \frac{\partial L}{\partial q_i^\cdot}\ (i = 1, \ldots, n), \quad \frac{\partial\Lambda}{\partial x_{n+1}} = t'\frac{\partial L}{\partial t} \tag{1.18}$$

$$p_{n+1} \equiv \frac{\partial\Lambda}{\partial x_{n+1}'} = L - \sum_{i=1}^{n} q_i^\cdot \frac{\partial L}{\partial q_i^\cdot} = -H(q_i, t, q_i^\cdot)$$

Clearly, the momentum vector of the system in variables x_α is equal to the momentum-energy vector in variables q_i.

Using the time t as the parameter τ and taking into account the relations (1.8), (1.18), we find that equations (1.17) for $\alpha = 1, \ldots, n$ take the form of the equations of motion (1.2), and for $\alpha = n+1$ the form of the energy equation

$$\frac{dH}{dt} + \frac{\partial L}{\partial t} = \sum_{l,i} \mu_l \frac{\partial f_l}{\partial q_i^\cdot} q_i^\cdot \tag{1.19}$$

which follows, as we know, from the equation of motion (1.2).

Thus we see that in the case of non-holonomic systems with constraints of the form (1.1), the dynamics based on the function $L(q, t, q^\cdot)$, or in short the L-dynamics, also represents a form of the Λ-dynamics in which the variable $x_{n+1} = t$ plays the part of a coordinate of the space R_{n+1}, as well as of the parameter on the curves $x_i = x_i(t)$ [7].

2. We shall now assume that the following conditions hold for the non-holonomic system in question:

$$\frac{\partial\Lambda}{\partial x_{n+1}} = \frac{\partial L}{\partial t} = 0, \quad \frac{\partial F_l}{\partial x_{n+1}'} = -\frac{1}{t'}\sum_{i=1}^{n} \frac{\partial f_l}{\partial q_i^\cdot} q_i^\cdot = 0 \tag{2.1}$$

These conditions mean that the functions $\Lambda(x_\alpha, x_\alpha')$ and $L(q, t, q^\cdot)$ do not depend explicitly on the variable $x_{n+1} = t$, while the functions $\partial F_l/\partial x_{n+1}'$ vanish by virtue of the connection equations (1.5), or in other words, the functions $f_l(q, q^\cdot)$ are homogeneous in $q_i^\cdot$. We assume for simplicity that the constraints (1.1) do not depend explicitly on t.

When conditions (2.1) hold, the last equation of (1.17), or equation (1.19), yields the following energy integral:

$$\frac{\partial \Lambda}{\partial x_{n+1}'} = -H(q, q^\cdot) = -h = \mathrm{const} \tag{2.2}$$

Here the coordinate $x_{n+1} = t$ is cyclic /10/ and the first integral of (2.2) corresponds to it. The function $\Lambda(x_l, x_\alpha')$ however depends explicitly on x_{n+1}'. This variable can be eliminated from the Hamilton principle (1.16) by using the energy integral (2.2) or the explicit expression

$$x_{n+1}' = t' = \varphi(q_i, q_i', h) \tag{2.3}$$

following from it, and replacing the variable x_{n+1}' in (1.16) by it, or by considering the integral (2.2) as a supplementary relationship /10/.

Note that the solution of (2.2) in the form (2.3) is impossible only in the exceptional case when the function $L(q, q^\cdot)$ is the sum of homogeneous functions of the zeroth and first degree in $q_i^\cdot$ /1/.

Using the first approach we introduce the Routh function

$$R(q_i, q_i', h) = \Lambda - \frac{\partial \Lambda}{\partial x_{n+1}'} x_{n+1}' = \Lambda + h x_{n+1}' = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i^\cdot} q_i' t' \tag{2.4}$$

We must replace, on the right-hand sides of these equivalent expressions, $x_{n+1}' = t'$ by (2.3). Replacing $\Lambda(x_\alpha, x_\alpha')$ by $R(q_i, q_i', h) - ht'$, in (1.16) we obtain the variational equation

$$\int_{\tau_0}^{\tau_1} \Delta R(q_i, q_i', h)\, d\tau = 0, \quad \Delta q_i = 0: \ \tau = \tau_0, \tau_1 \tag{2.5}$$

which expresses the principle of least action in Jacobi form, where the constant h has a single fixed value for all comparison curves.

For example, in the case of an ordinary non-homonomic dynamic system for which the Lagrange function is

$$L(q, q^\cdot) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(q) q_i^\cdot q_j^\cdot + \sum_{i=1}^{n} a_i(q) q_i^\cdot + L_0(q) \tag{2.6}$$

Eq.(2.5) will take the well-known form

$$\int_{\tau_0}^{\tau_1} \Delta\Big(\sqrt{2(h + L_0)} \sqrt{\sum_{i,j=1}^{n} a_{ij} q_i' q_j'} + \sum_{i=1}^{n} a_i q_i'\Big) d\tau = 0, \quad \Delta q_i = 0: \ \tau = \tau_0, \tau_1$$

The principle (2.5), taking (1.9) and (2.1) into account, yields the differential equations for the real trajectories of the system

$$\frac{d}{d\tau} \frac{\partial R}{\partial q_i'} - \frac{\partial R}{\partial q_i} = \sum_{l} \mu_l \frac{\partial F_l}{\partial q_i'} \quad (i = 1, \ldots, n) \tag{2.7}$$

which must be supplemented by the constraint equations (1.5). By integrating the equations (2.7), (1.5), we can obtain the parameteric equations of the trajectory of the real motion of the system in the configurational space R_n, not containing time. The time dependence can be found by integrating (2.3), which thus complements the variational equation (2.5). Taking into account the relations $R = (L + h) t'$ and replacing the parameter τ by t we find, that according to (2.3) equations (2.7) take the form of the equations of motion (1.2).

Using the energy integral (2.2) we can reduce the initial Lagrange system with constraints, as in the case of a holonomic system /11/, to a Lagrangian system with a reduced number of degrees of freedom. Indeed, let us take one of the Lagrange coordinates as the parameter τ e.g. q_1, and express the quantity $q_1^\cdot$ appearing in the energy integral in the form

$$q_1^\cdot = \frac{1}{t'} = \psi(q_i, q_s', h) \tag{2.8}$$

where $q_s' = dq_s/dq_1$, $t' = dt/dq_1$, and $q_s^\cdot = q_1^\cdot q_s'$ $(s = 2, \ldots, n)$. Substituting (2.8) into (2.4) and (1.1), we obtain the following expression for the Routh function:

$$R(q_i, q_s', h) = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i^{\cdot}} \frac{q_i^{\cdot}}{q_1^{\cdot}} \tag{2.9}$$

and the constraint equation $f_l(q_i, q_i') = 0$ $(l = 1, \ldots, r)$ in the variables q_i, q_i'. Here the principle of least action in the Jacobi form (2.5) takes the form

$$\int_{q_1^0}^{q_1^1} \delta R(q_i, q_s', h)\, dq_1 = 0, \quad \delta q_s = 0: \; q_1 = q_1^0, q_1^1 \tag{2.10}$$

where q_1^0, q_1^1 are the values of the variable q_1 for the initial and final position of the system, and the variations δq_s satisfy the conditions

$$\sum_{s=2}^{n} \frac{\partial f_l}{\partial q_s'} \delta q_s = 0 \quad (l = 1, \ldots, r)$$

into which conditions (1.4) transform, provided that we remember that in the present case we should write /12/ $\delta q_1 = 0$, and take into account conditions (1.13) and (1.10).

Principle (2.10) yields the equations of motion of the system

$$\frac{d}{dq_1}\left(\frac{\partial R}{\partial q_s'}\right) - \frac{\partial R}{\partial q_s} = \sum_l \mu_l \frac{\partial f_l}{\partial q_s'} \quad (s = 2, \ldots, n) \tag{2.11}$$

representing the Jacobi /12/-Whittaker /11/ equations generalized to include non-holonomic systems.

In the case of an ordinary non-holonomic dynamic system with Lagrange function (2.6), the Routh function is

$$R(q_i, q_s', h) = 2\sqrt{(h + L_0) G} + \Phi \tag{2.12}$$

where

$$G(q_i, q_i') = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(q) q_i' q_j'$$

$$\Phi(q_i, q_i') = \sum_{i=1}^{n} a_i(q) q_i', \; \psi(q_i, q_s', h) = \sqrt{\frac{h + L_0}{G}}$$

We note that (2.10) has the form of the Hamilton principle. It follows that the principle of least action in Jacobi form (2.5) for a non-holonomic system, with constraints (1.1) and the Lagrange function $L(q, q^{\cdot})$ for which the energy integral (2.2) exists, is identical with the Hamilton principle (2.10) for a reduced system with constraints $f_l(q_i, q_i') = 0$ and Lagrange function $R(q_i, q_s', h)$ /11/.

In the second approach, when the integral (2.2) is regarded as a relationship supplementing the variational equation, we replace the function $\Lambda(x_\alpha, x_\alpha')$ in (1.16) by its expression

$$\Lambda = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i'} q_i' - ht'$$

which follows from (2.4). As a result we obtain the new variational equation

$$\Delta \int_{\tau_0}^{\tau_1} \sum_{i=1}^{n} \frac{\partial \Lambda}{\partial q_i'} q_i'\, d\tau = 0, \quad \Delta q_i = 0: \; \tau = \tau_0, \tau_1 \tag{2.13}$$

with the additional relation (2.2) in which the constant h has a single fixed value for all comparison curves. Equation (2.13) expresses the principle of least action in Lagrange form written in parametric form. If we take the time t as the parameter τ, equation (2.13) will take the well-known form of the Lagrange principle in the space R_n

$$\Delta \int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{\partial L}{\partial q_i^{\cdot}} q_i^{\cdot}\, dt = 0, \quad \Delta q_i = 0: \; t = t_0, t_1 \tag{2.14}$$

taking conditions (2.2) and (1.10), and (1.13) into account. The upper limit t_1 in (2.14) is not fixed, but depends on the comparison curve.

3. We shall consider a system of material points acted upon by potential forces with force function $U(r_\nu)$ and constrained by the following perfect, finite non-stationary

constraints:

$$\varphi_s(\mathbf{r}_\nu, t) = 0 \quad (s = 1, \ldots, k) \tag{3.1}$$

and non-integrable constraints homogeneous in $r_\nu^\cdot = dr_\nu/dt$. Here $\mathbf{r}_\nu$ $(\nu = 1, \ldots, N)$ are the radius vectors of the points of the system relative to the origin of the inertial coordinate system which can be expressed, after introducing the Lagrange coordinates q_i $(i = 1, \ldots, n = 3N - k)$, in the form of the functions

$$\mathbf{r}_\nu = \mathbf{r}_\nu(q_1, \ldots, q_n, t) \quad (\nu = 1, \ldots, N)$$

Here the constraints (3.1) are satisfied identically and the non-integrable constraints take the form (1.1). The Lagrange function is $L(q, t, q^\cdot) = T + U$ where $T(q, t, q^\cdot)$ is the kinetic energy, and the force function $U(q, t)$ in this case has the form (2.6), i.e. $L = L_2 + L_1 + L_0$ and

$$L_2 = T_2 = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} q_i^\cdot q_j^\cdot, \quad L_1 = T_1 = \sum_{i=1}^{n} a_i q_i^\cdot, \quad L_0 = T_0 + U$$

The equations of motion (1.2), (1.1) lead to an equation of the form (1.19) expressing the generalized energy theorem

$$d(T_2 - T_0 - U) = -\frac{\partial L}{\partial t}dt + \sum_{l,i}\mu_l \frac{\partial f_l}{\partial q^\cdot} dq_i \tag{3.2}$$

since in the present case we have

$$H(q, t, q^\cdot) = L_2 - L_0 = T_2 - T_0 - U$$

On the other hand, according to the general theorem on kinetic energy we have the following differential equation:

$$dT = dU + \sum_\nu \mathbf{R}_\nu \cdot d\mathbf{r}_\nu \tag{3.3}$$

where R_ν are the reactions of the constraints (3.1). Clearly, in the case of non-stationary constraints (3.1), equation (3.2) differs from (3.3), i.e. only the equations (1.2), (1.1) prevent us from obtaining the general theorem on kinetic energy. This is understandable, since equations (1.2) do not contain the reactions of the constraints (3.1) which provide a contribution to the energy change according to (3.3).

Passing now to the coordinates q_i and integrating both sides of (3.3), with respect to time, we obtain the well-known theorem on energy in its final form

$$T(q, t, q^\cdot) - U(q, t) = \int_{t_0}^{t}\sum_\nu \mathbf{R}_\nu \cdot d\mathbf{r}_\nu + \text{const} \tag{3.4}$$

The integral on the right-hand side of (3.4) can generally be found only after integrating the system of differential equations of motion; therefore, relation (3.4) expresses in general only what it represents, namely the relation between the energy and the work of the reactions of constraints /13/. However, when the perfect constraints (3.1) do not depend explicitly on time, $\sum_\nu \mathbf{R}_\nu \cdot d\mathbf{r}_\nu = 0$, and (3.4) becomes the first integral

$$T(q, q^\cdot) - U(q) = \text{const} \tag{3.5}$$

which also follows from (3.2) since in this case we have

$$T = T_2, \quad \partial L/\partial t = 0 \quad \sum_i \frac{\partial f_l}{\partial q_i^\cdot} q_i^\cdot = 0.$$

When the constraints (3.1) are non-stationary, cases are possible when the function L does not depend explicitly on time. In these cases, if

$$\sum_i \frac{\partial f_l}{\partial q_i^\cdot} q_i^\cdot = 0,$$

then equation (3.2) yields the first integral of the form (2.2)

$$T_2(q, q^\cdot) - T_0(q) - U(q) = h$$

which however does not follow directly from (3.3).

Combining (3.2) and (3.3), we find the expression for the work done by the reactions of the constraints (3.1) over the real displacement of the system

$$\sum_\nu \mathbf{R}_\nu \cdot d\mathbf{r}_\nu = d(T_1 + 2T_0) - \frac{\partial L}{\partial t}dt + \sum_{l,i}\mu_l \frac{\partial f_l}{\partial q_i^\cdot} dq_i \tag{3.6}$$

obtained earlier /14/ for the case of linear homogeneous constraints (1.1). Since the constraints (3.1) are assumed to be perfect,

$$\sum_{\nu} \mathbf{R}_\nu \cdot d\mathbf{r}_\nu = \sum_{\nu} \mathbf{R}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial t} dt = R_0 \, dt$$

where R_0 is the generalized reaction of the non-stationary constraints (3.1) corresponding to the coordinate $t = q_0$ and representing the strength of the non-stationary constraints (3.1). If we assume that $q_0^{\cdot} = 1$, the kinetic energy can be written as the quadratic form /15/

$$2T = \sum_{\alpha, \beta=0}^{n} a_{\alpha\beta}(q) \, q_\alpha^{\cdot} q_\beta^{\cdot}$$

where $a_{i0} = a_i(q)$, $a_{00} = 2T_0$, and we obtain the relation

$$\frac{\partial T}{\partial q_0^{\cdot}} = \sum_{i=1}^{n} a_{i0} q_i^{\cdot} + a_{00} q_0^{\cdot} = T_1 + 2T_0$$

Dividing both sides of (3.6) by dt, we obtain a Lagrange-type equation for the coordinate $t = q_0$ of the non-holonomic system

$$\frac{d}{dt} \frac{\partial L}{\partial q_0^{\cdot}} - \frac{\partial L}{\partial q_0} = R_0 - \sum_{l, i} \mu_l \frac{\partial f_l}{\partial q_i^{\cdot}} q_i^{\cdot} \tag{3.7}$$

The above equation can be derived independently of (3.2), (3.3), by projecting the Newton equation on to the direction $\partial \mathbf{r}_\nu / \partial t$ and summing over ν. This is how an equation of the form (3.7) was obtained in /15/ for a holonomic system.

Equation (3.7) can be used to determine the intensity R_0 of reactions of the non-stationary constraints (3.1). In the case of a single constraint it enables us to determine its reaction after integrating (1.2), (1.1). We note that in /16/ a method is also given for determining the reaction of non-stationary constraints by projecting on to the direction of the vector $\partial \mathbf{r}_\nu / \partial t$.

Using the Lagrange equations for q_i and t, an equation of the type (3.4) was obtained in /15/, which was called the energy integral. However the equation contains an a priori unknown quantity R_0 and, as was said before, cannot be used as the first integral.

We find, however, that supplementing the equations (1.2), (1.1), with (3.7) obtained independently of (3.2) and (3.3), is useful not only in determining R_0, but also in deriving from these equations the theorem on kinetic energy (3.3). To do this it is sufficient to combine (3.2) and (3.6), or to obtain (3.2) from (3.3) it is sufficient to subtract from (3.3) equation (3.6), which is equivalent to equation (3.7).

REFERENCES

1. LEVI-CIVITA T. and AMALDI U., Compendio di meccanico razionale. Bologna, Zanichelli, 1948.
2. CHETAEV N.G., On Gauss's principle. Kazan', Izv. fiz.-matem. ob-va pri Kazan. un-te, Vol.6, No.3, 1932-1933.
3. RUMYANTSEV V.V., On Hamilton's principle for non-holonomic systems. PMM Vol.42, No.3, 1978.
4. WEBER R.W., Kanonische Theorie nichtholonomer systeme. Peter Lang - Bern, 1981.
5. WEBER R.W., On Hamiltonian systems with non-holonmic constraints. Proceedings of the IUTAM-ISIMM Symposium. Modern developments in analytical mechanics. Acad. Sci. of Turin, Vol.1, 1983.
6. GEL'FAND I.M. and FOMIN S.V., Variational Calculus. Moscow, Fizmatgiz, 1961.
7. SING D.L., Classical Dynamics. Moscow, Fizmatgiz, 1963.
8. RUMYANTSEV V.V., On some problems of analytical dynamics of non-holonomic systems. Proceedings of the IUTAM-ISIMM symposium. Modern developments in analytical mechanics. Acad. Sci. of Turin Vol.2, 1983.
9. PARS L., Analytical Dynamics. Moscow, Nauka, 1971.
10. LANCZOS C., The Variational Principles of Mechanics. Toronto, Univ. of Toronto Press, 1970.
11. WHITTAKER E.T., Analytical Dynamics. Cambridge, N.Y., 1937.
12. JACOBI K.G.J., Vorlesungen über dynamik. Berlin 1866.
13. LYAPUNOV A.M., Lectures on Theoretical Mechanics. Kiev. Nauk, dumka, 1982.
14. RUMYANTSEV V.V., On the kinetic energy theorem. Vestn. MGU, Matem. mekhan. No.3, 1967.
15. BUIICHICH V.A., On the energy integral of svstems with non-stationary constraints. Teoretich i prikl. mekhanika, Vol.6, Beograd, 1980.
16. SIL'DE O. and REL'VIK KH., Equation of possible power in analytical mechanics. Izv. Akad. Nauk EstSSR, Vol.32, No.4, 1983.

Translated by L.K.